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Evaluating the State Probabilities of *M* out of *N* Sliding Window Detectors

Peter Williams

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Peter Williams

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ABSTRACT

The term *M out of N sliding window detectors* refers to processes that determine whether there have been M successes in a sequence of N binary trials, where the *window* (of N trials under examination) *slides* along a possibly infinite sequence stopping when the criterion (of M successes in N successive trials) is met. They are frequently used to model the operators of naval surveillance systems such as radar and sonar. When an M out of N sliding window detector is examining a sequence of trials it may be in one of several states. The state of most interest is the *accepting* state into which it enters (and remains) when it encounters a sequence of N successive trials containing M successes. This paper describes a generalised method for estimating the *probability* that an M out of N sliding window detector is in its accepting state given a sequence of probabilities representing the likelihood of success on each of a sequence of binary trials.

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Evaluating the state probabilities of M out of N sliding window detectors

EXECUTIVE SUMMARY

The term *M out of N sliding window detectors* refers to processes that determine whether there have been M successes in a sequence of N binary trials, where the *window* (of N trials under examination) *slides* along a possibly infinite sequence stopping when the criterion (of M successes in N successive trials) is met. They are frequently used to model the operators of naval surveillance systems such as radar and sonar. This is because these systems are subject to false alarms and the operators use the persistence (scan-to-scan in the case of radar and ping-to-ping in the case of sonar) of the target echoes (as one cue) in distinguishing them from false alarms. As the echoes are intermittent (especially when the target is at extreme ranges), the operators are quite tolerant of missing echoes and might, for instance, only require that the echo appears on two out of three consecutive scans/pings (in which case they would be performing as a *2 out of 3 sliding window detector*).

When an M out of N sliding window detector is examining a sequence of trials it may be in one of several states. The state of most interest is the *accepting* state which it enters when it encounters a sequence of N successive trials containing M successes and stays there. All other states that an M out of N sliding window detector may enter while examining a sequence of trials are known as *non-accepting* states and this includes the *initial* state which is the state that the detector is in before any trials have been evaluated.

The implementation of M out of N sliding window detectors is quite simple and their incorporation into simulations to model sensor operators is straightforward. Within the simulation the appearances (or otherwise) of the echoes is determined and the result is fed to a simulation of the required sliding window detector which will declare a detection when the criterion is met and enter the accepting state.

However, in studies of a more theoretical nature, what is known about the series of trials (e.g. radar sweeps or sonar pings) is not the result (positive or negative) of each trial but, rather, the probability that the trial would return a positive result. In these cases, what is required is an estimate of the *probability* that the sliding window detector is in its accepting state. A generalised (and efficient) method for solving this problem (for arbitrary values of M and N) is developed in this paper based on finite state automata. The solution is simple and easily implemented in modern computer languages.

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Peter Williams is a Senior Professional Officer (Grade C) working within the Maritime Operations Division at Pyrmont. He retired from the Royal Australian Navy in 1990 after 25 years service as a naval aircrew officer. While in the navy he gained considerable experience in carrier borne fixed wing antisubmarine warfare both as a tactical coordinator and sensor operator. He served as Tactical Instructor (Air) at the RAN Tactical School from 1975 until 1979. He was awarded a first class honours degree in Computer Science by Sydney University in 1987 and a Doctor of Philosophy in 1996 for his thesis, *Dynamic Memory for Design*, by the same university.

Contents

1 Introduction **1**

2 Implementing M out of N sliding window detectors **1**

3 Evaluating M out of N sliding window detectors **4**

4 A General Method of Evaluation **6**

5 A General and Optimal Method of Evaluation **7**

6 Example of Application **11**

7 Conclusion **13**

8 References **14**

Figures

1 State transition diagram for 3 out of 4 sliding window detector 4

2 State transition diagram for an optimal 3 out of 4 sliding window detector . . 8

3 The probability, p, of detection on a single ping as a function of range 12

4 The probabilities, $p_\star$, that satisfaction of the 3 blips in 5 pings criterion is met at a given range and, $P_\star$, that satisfaction of the criterion occurs by a given range 13

Tables

1 State transition table for 3 out of 4 sliding window detector 3

2 State transition table for an optimal 3 out of 4 sliding window detector 8

3 State transition tables for 5 out of 6 sliding window detectors 9

Algorithms

1 Conduct binary trials until M successes are achieved in N successive trials . . 2

2 Evaluate M out of N sliding window detector 6

3 Create the set of hexads $\bar{\mathrm{S}}_{M,N}$ 6
4 Add a hexad with $H = i$ to $\bar{\mathrm{S}}_{M,N}$ 7
5 Create the optimal set of hexads $\bar{\mathrm{S}}'_{M,N}$ 10
6 Add a hexad with $H = i$ to the optimal set $\bar{\mathrm{S}}'_{M,N}$ 11
7 Smallest equivalent history to i in an M of N automaton 11

1 Introduction

The term *M out of N sliding window detectors* refers to processes that determine whether there have been M successes in a sequence of N binary trials, where the *window* (of N trials under examination) *slides* along a possibly infinite sequence stopping when the criterion (of M successes in N successive trials) is met. They are frequently used to model the operators of naval surveillance systems such as radar and sonar (refs 1 and 2). This is because these systems are subject to false alarms and the operators use the persistence (scan-to-scan in the case of radar and ping-to-ping in the case of sonar) of the target echoes (as one cue) in distinguishing them from false alarms. As the echoes are intermittent (especially when the target is at extreme ranges), the operators are quite tolerant of missing echoes and might, for instance, only require that the echo appears on two out of three consecutive scans/pings (in which case they would be performing as a *2 out of 3 sliding window detector*).

When an M out of N sliding window detector is examining a sequence of trials it may be in one of several states. The state of most interest is the *accepting* state which it enters when it encounters a sequence of N successive trials containing M successes and stays there. All other states that an M out of N sliding window detector may enter while examining a sequence of trials are known as *non-accepting* states and this includes the *initial* state which is the state that the detector is in before any trials have been evaluated.

As will be shown in Section 2 the implementation of M out of N sliding window detectors is quite simple and their incorporation into simulations to model sensor operators is straightforward. Within the simulation the appearances (or otherwise) of the echoes is determined and the result is fed to a simulation of the required sliding window detector which will declare a detection when the criterion is met and enter the accepting state.

However, in studies of a more theoretical nature, what is known about the series of trials (e.g. radar sweeps or sonar pings) is not the result (positive or negative) of each trial but, rather, the probability that the trial would return a positive result. In these cases, what is required is an estimate of the *probability* that the sliding window detector is in its accepting state. The topic of this paper is the solution of this problem. The key to this problem of evaluating the state probabilities of M out of N sliding window detectors lies in examining the methods for implementing them and to this end the next section will consider their implementation.

2 Implementing M out of N sliding window detectors

In order to implement an M out of N sliding window detector it is necessary to maintain a *history* of the last $N-1$ trials. If the result of each trial is represented by a single binary digit, with 1 representing a positive trial result and 0 a negative one, then

the relevant history H after i trials is given by the string (of length $N-1$) of binary digits

$$H = R_{i-N+2}R_{i-N+3}\ldots R_{i-1}R_i \tag{1}$$

where R_j is the binary digit representing the result of the jth trial. For example, if $N = 4$, and the result of the last three $(N-1)$ trials was positive, negative and negative (in that order) then the history is $H = 100$. In order to determine if the M successful trials out of N consecutive trials criterion has been satisfied after the completion of a trial with a positive result (given that the criterion has not already been satisfied) it is necessary to determine the number of digits $\#(H)$ with the value 1 in the history H and if it equals $M-1$ the criterion is satisfied. If the criterion was not satisfied (after a successful trial) it is necessary to update the history H in preparation for the next trial and this entails deleting the left most digit from H and adding a 1 to the right. For example, if $H = 100$ before a successful trial that fails to satisfy the criterion then it becomes $H = 001$ after the trial. If H is considered to be the binary representation of an integer number this is equivalent to the integer arithmetic operation $H \Leftarrow (H \times 2) \ \mathbf{mod}\ 2^{N-1} + 1$ where $x \Leftarrow y$ means that x is given the value y and $i \ \mathbf{mod}\ j$ is the modulus of i with respect to j.

In the event of a negative result to a trial it is not necessary to check for the satisfaction of the M out of N criterion as, if it were satisfied, it would have been previously determined. All that is necessary is to adjust the history H in preparation for the consideration of the next trial. This necessitates the removal of the left most digit and the addition of a 0 to the right end. If H is considered to be the binary representation of an integer number this is equivalent to the integer arithmetic operation $H \Leftarrow (H \times 2) \ \mathbf{mod}\ 2^{N-1}$.

Algorithm 1 Conduct binary trials until M successes are achieved in N successive trials

```
accept ⇐ false
H ⇐ 0
repeat
  carry out binary trial
  if the result of the binary trial is positive then
    if #(H) = (M − 1) then {M out of N criterion is satisfied}
      accept ⇐ true
    else
      H ⇐ (H × 2) mod 2^(N−1) + 1
    end if
  else
    H ⇐ (H × 2) mod 2^(N−1)
  end if
until accept = true
```

Algorithm 1 gathers the ideas presented above to provide a method for conducting a series of binary trials until the criterion of M successes out of N successive trials is satisfied. Another view that can be taken of this process is that it is a finite state automaton that transitions from one state to another dependent on the result of a binary trial until it eventually enters the *accepting* state. In this view of the process, each possible value of the history H can be considered as a separate state of the automaton. Let $S_{M,N}$ be the set of such states that an M out of N sliding window may enter. In addition to the *accepting*

state there will be one state in this set for each integer that can be represented by $N-1$ binary digits minus those integers with $N-1$ digits that contain M or more digits with the value 1 and the number of states $^{\#}S_{M,N}$ in the set can be shown (ref. 3) to be given by

$$^{\#}S_{M,N} = \begin{cases} \sum_{k=0}^{N-M} T_k & \text{when } M \leq N \leq 2M-1 \\ \sum_{k=0}^{M-1} T_k & \text{when } N \geq 2M-1 \end{cases} \tag{2}$$

where

$$T_k = \begin{cases} 2^{M-1} & \text{when } k = 0 \\ \left[2^{M-1} - \sum_{i=0}^{k-1} \binom{M}{i}\right] \binom{N-M}{k} & \text{when } k > 0 \end{cases} \tag{3}$$

In order to demonstrate this concept consider the case where $M = 3$ and $N = 4$. There are 8 integers (0, 1, 2, 3, 4, 5, 6 and 7) that can be represented by 3 (i.e. $N-1$) binary digits and one of these (7) contains 3 (i.e. M) digits with the value 1. Let the label of each non-accepting state be the integer interpretation of H that is extant when the system is in that state and let the label of the accepting state be the $\star$ symbol. Then, using state transitions that are determined by the processes described in Algorithm 1, the finite state automaton that implements a 3 out of 4 sliding window detector is described by the state transition table shown in Table 1. Alternatively, it is described by the state transition diagram shown in Figure 1 where the nodes represent the various states and the edges represent the transitions. It is this view of the system as a finite state automaton that

From State	To State	
	−ve	+ve
0	0	1
1	2	3
2	4	5
3	6	$\star$
4	0	1
5	2	$\star$
6	4	$\star$
$\star$	$\star$	$\star$

Table 1: State transition table for 3 out of 4 sliding window detector

is the basis for the evaluation of M out of N sliding window detectors when the actual results of each binary trial are not known but, instead, the probabilities that each of a sequence of binary trials is successful is known.

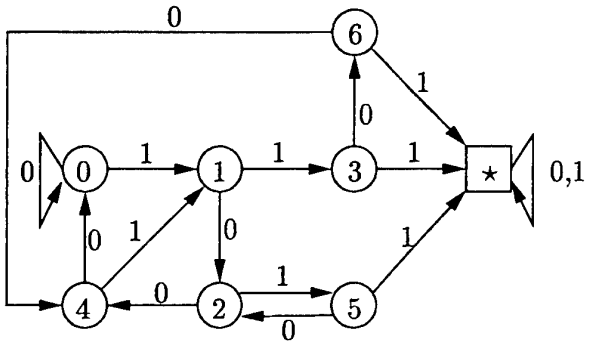

Figure 1: State transition diagram for 3 out of 4 sliding window detector

3 Evaluating M out of N sliding window detectors

If the notation $P_s(k)$ is used to represent the probability that the process is in state s after the kth trial then the probability that the process is in the accepting state after the kth trial is given by $P_\star(k)$. In order for the system to be in state s after the kth trial either the kth trial must have had a positive result and the system was in a state where it transitions to the state s after a positive result or the kth trial had a negative result and the system was in a state where it transitions to state s on a negative result and, therefore,

$$P_s(k) = p_k P_s^+(k-1) + (1-p_k) P_s^-(k-1) \tag{4}$$

where p_k is the probability that the kth trial has a positive result, $P_s^+(k-1)$ is the probability that the system is in a state, after the $(k-1)$th trial, that will transition to state s if the kth trial has a positive result and $P_s^-(k-1)$ is the probability that the system is in a state, after the $(k-1)$th trial, that will transition to state s if the kth trial has a negative result.

For example, in the case where $M = 3$ and $N = 4$ the positive transition probabilities can be determined to be

$$\begin{aligned}
P_0^+(k-1) &= 0 && (5)\\
P_1^+(k-1) &= P_0(k-1) + P_4(k-1) && (6)\\
P_2^+(k-1) &= 0 && (7)\\
P_3^+(k-1) &= P_1(k-1) && (8)\\
P_4^+(k-1) &= 0 && (9)\\
P_5^+(k-1) &= P_2(k-1) && (10)\\
P_6^+(k-1) &= 0 && (11)\\
P_\star^+(k-1) &= P_3(k-1) + P_5(k-1) + P_6(k-1) + P_\star(k-1) && (12)
\end{aligned}$$

from Table 1 and/or Figure 1. Similarly the negative transition probabilities can be

determined to be

$$\begin{aligned}
P_0^-(k-1) &= P_0(k-1) + P_4(k-1) && (13)\\
P_1^-(k-1) &= 0 && (14)\\
P_2^-(k-1) &= P_1(k-1) + P_5(k-1) && (15)\\
P_3^-(k-1) &= 0 && (16)\\
P_4^-(k-1) &= P_2(k-1) + P_6(k-1) && (17)\\
P_5^-(k-1) &= 0 && (18)\\
P_6^-(k-1) &= P_3(k-1) && (19)\\
P_\star^-(k-1) &= P_\star(k-1) && (20)
\end{aligned}$$

and this, in turn, allows the full set of state probability equations for a 3 out of 4 sliding window detector to be determined

$$\begin{aligned}
P_0(k) &= (1-p_k)(P_0(k-1) + P_4(k-1)) && (21)\\
P_1(k) &= p_k(P_0(k-1) + P_4(k-1)) && (22)\\
P_2(k) &= (1-p_k)(P_1(k-1) + P_5(k-1)) && (23)\\
P_3(k) &= p_k P_1(k-1) && (24)\\
P_4(k) &= (1-p_k)(P_2(k-1) + P_6(k-1)) && (25)\\
P_5(k) &= p_k P_2(k-1) && (26)\\
P_6(k) &= (1-p_k) P_3(k-1) && (27)\\
P_\star(k) &= P_\star(k-1) + p_k(P_3(k-1) + P_5(k-1) + P_6(k-1)) && (28)
\end{aligned}$$

which can be used to evaluate the detector when started with the following initial values:

$$P_s(0) = \begin{cases} 1 & \text{when } s = 0 \\ 0 & \text{otherwise.} \end{cases} \quad (29)$$

As the detector remains in the accepting state once it has entered that state the probability $p_\star(k)$ that it enters the accepting state as a result of the kth trial is given by

$$p_\star(k) = \begin{cases} P_\star(k) & \text{when } 0 \le k \le 1 \\ P_\star(k) - P_\star(k-1) & \text{otherwise.} \end{cases} \quad (30)$$

and this is equivalent to the probability density function for $P_\star(k)$. It can then be used to determine the most probable detection range(s). The set of equations needed to evaluate M out of N sliding window detectors with other values for M and N can be obtained by similar processes to that just described.

Although they will correctly evaluate M out of N sliding window detectors, the methods just outlined suffer from one severe disadvantage. Every time it is required to evaluate a sliding window detector for values of M and N that have not been previously encountered, a new set of equations must be generated. What is required is a more general method that can automatically accommodate many combinations of M and N and this will be developed in the next section.

4 A General Method of Evaluation

In order to generalise the methodology described in the previous section each possible state of a system will be represented by an ordered hexad $(H, P, H^+, H^-, P^+, P^-)$, where H is the states identifying number, P is the probability that the system is this state, H^+ is the identity of the state reached from this state after a positive trial result, H^- is the identity of the state reached from this state after a negative trial result and P^+ and P^- are provided to hold intermediate values during the evaluation process. The set $\bar{\mathrm{S}}_{M,N}$ is the set of such hexads necessary to represent all of the states of an M out of N sliding window detector and the notation $\bar{\mathrm{S}}_{M,N}[j]$ is used to refer to the member of $\bar{\mathrm{S}}_{M,N}$ that has $H = j$. Where necessary, dot notation will be used to signify components of a hexad (e.g. the notation $\bar{\mathrm{S}}_{M,N}[j].P$ refers to the P component of the hexad $\bar{\mathrm{S}}_{M,N}[j]$).

Algorithm 2 Evaluate M out of N sliding window detector

$\bar{\mathrm{S}}_{M,N}[0].P \Leftarrow 1$
for each hexad h in $\bar{\mathrm{S}}_{M,N}$ where $h.H \neq 0$ **do**
  $h.P \Leftarrow 0$
end for
loop
  determine the probability p of a positive result on the current trial
  for each hexad h in $\bar{\mathrm{S}}_{M,N}$ **do**
    $h.P^+ \Leftarrow 0$
    $h.P^- \Leftarrow 0$
  end for
  for each hexad h in $\bar{\mathrm{S}}_{M,N}$ **do**
    $\bar{\mathrm{S}}_{M,N}[h.H^+].P^+ \Leftarrow \bar{\mathrm{S}}_{M,N}[h.H^+].P^+ + h.P$
    $\bar{\mathrm{S}}_{M,N}[h.H^-].P^- \Leftarrow \bar{\mathrm{S}}_{M,N}[h.H^-].P^- + h.P$
  end for
  for each hexad h in $\bar{\mathrm{S}}_{M,N}$ **do**
    $h.P \Leftarrow p\,(h.P^+) + (1-p)\,(h.P^-)$
  end for
  wait for next trial
end loop

Algorithm 2 illustrates a process by which the set of hexads $\bar{\mathrm{S}}_{M,N}$ may be used to evaluate an M out of N sliding window detector. After initialising each hexad's state probability P to the appropriate values, it enters an infinite loop and evaluates each of the state probabilities once the probability of a positive result from the current trial is known. At any time that the system is waiting for the next trial it may be interrogated to determine the various state probabilities. In particular, the value of $\bar{\mathrm{S}}_{M,N}[\star].P$ will be the probability that the system has successfully met the M out of N criterion based on the trials processed so far and is, therefore, in the accepting state.

Algorithm 3 Create the set of hexads $\bar{\mathrm{S}}_{M,N}$

$\bar{\mathrm{S}}_{M,N} = \{(\star, 0, \star, \star, 0, 0)\}$
Add a hexad with $H = 0$ to $\bar{\mathrm{S}}_{M,N}$ – **Algorithm 4**
$\bar{\mathrm{S}}_{M,N}[0].P \Leftarrow 1$

In order for Algorithm 2 to be applied it is first necessary to generate the set of hexads $\bar{\mathrm{S}}_{M,N}$ and Algorithm 1 provides the key to solving this problem. Based on the information in that algorithm, the recursive algorithm, Algorithm 4, and Algorithm 3 can be used to construct the set $\bar{\mathrm{S}}_{M,N}$. Algorithm 3 initialises $\bar{\mathrm{S}}_{M,N}$ to contain a single hexad representing the accepting state and then uses Algorithm 4 to add a hexad for the initial state $H = 0$ to $\bar{\mathrm{S}}_{M,N}$.

Algorithm 4 Add a hexad with $H = i$ to $\bar{\mathrm{S}}_{M,N}$

```
create new hexad h
h.H ⇐ i
h.P ⇐ 0
if #(i) = (M − 1) then
  h.H+ ⇐ ⋆
else
  h.H+ ⇐ (i × 2) mod 2^(N−1) + 1
end if
h.H− ⇐ (i × 2) mod 2^(N−1)
S̄M,N ⇐ S̄M,N ∪ {h}
if a hexad with H = h.H+ does not exist in S̄M,N then
  Add a hexad with H = h.H+ to S̄M,N  − Algorithm 4
end if
if a hexad with H = h.H− does not exist in S̄M,N then
  Add a hexad with H = h.H− to S̄M,N  − Algorithm 4
end if
```

The recursive nature of Algorithm 4 then ensures that hexads for all of the other states that an M out of N detector requires are added to $\bar{\mathrm{S}}_{M,N}$ with their state probabilities set to 0. Finally Algorithm 3 sets the state probability for state 0 to 1 ensuring that $\bar{\mathrm{S}}_{M,N}$ is in a valid configuration (with the sum of the state probabilities equal to 1). The number $^{\#}\bar{\mathrm{S}}_{M,N}$ of hexads in the set $\bar{\mathrm{S}}_{M,N}$ generated by these algorithms is given by

$$^{\#}\bar{\mathrm{S}}_{M,N} = {}^{\#}\mathrm{S}_{M,N} \tag{31}$$

(see Equation 2) and may be quite large.

5 A General and Optimal Method of Evaluation

From Algorithm 2, it can be readily determined that the time taken to evaluate the system each time a trial is processed will be proportional to the number $^{\#}\bar{\mathrm{S}}_{M,N}$ of hexads in $\bar{\mathrm{S}}_{M,N}$ (which is given by Equation 2 and can be very large depending on the values of M and N). Therefore, if it is possible to reduce the number of hexads in $\bar{\mathrm{S}}_{M,N}$, without altering the value obtained for $\bar{\mathrm{S}}_{M,N}[\star].P$ after each trial is processed, then the evaluation process can be made more efficient. That this is possible, at least in the case where $M = 3$ and $N = 4$, can be determined by examining Table 1 and Figure 1 and observing that two states are *effectively* equivalent if their transition states are identical. In the case where $M = 3$ and $N = 4$ the states 0 and 4 both transition to state 0 on a negative trial

result and to state 1 on a positive trial result. They are, therefore, effectively equivalent and can be merged into a single state to form a new finite state automaton, described in Table 2 and Figure 2, that is also capable of determining when there have been 3 trials

From State	To State	
	−ve	+ve
0	0	1
1	2	3
2	0	5
3	6	⋆
5	2	⋆
6	0	⋆
⋆	⋆	⋆

Table 2: State transition table for an optimal 3 out of 4 sliding window detector

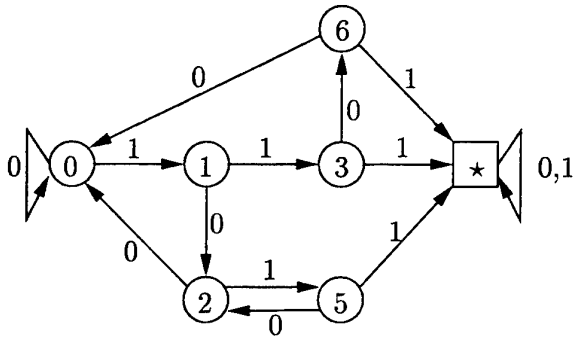

Figure 2: State transition diagram for an optimal 3 out of 4 sliding window detector

with positive results out of 4 successive trials. Additionally, an equivalent set of equations to Equations 21 to 28

$$P_0(k) = (1-p_k)(P_0(k-1) + P_2(k-1) + P_6(k-1)) \quad (32)$$
$$P_1(k) = p_k(P_0(k-1)) \quad (33)$$
$$P_2(k) = (1-p_k)(P_1(k-1) + P_5(k-1)) \quad (34)$$
$$P_3(k) = p_k P_1(k-1) \quad (35)$$
$$P_5(k) = p_k P_2(k-1) \quad (36)$$
$$P_6(k) = (1-p_k) P_3(k-1) \quad (37)$$
$$P_\star(k) = P_\star(k-1) + p_k(P_3(k-1) + P_5(k-1) + P_6(k-1)) \quad (38)$$

could be generated from this new automaton in order to evaluate 3 out of 4 sliding window detectors (if desired).

In general, the process of reducing a finite state automaton to its optimal form will be more complex than this example and several iterations may be required. Take, for

example, the case where $M = 5$ and $N = 6$. The finite state automaton generated using the methods described in Section 2 is given in Table 3(a) and contains 31 states. Among these states there are 11 pairs of equivalent states (16 and 0, 17 and 1, 18 and 2, 19 and 3, 20 and 4, 21 and 5, 22 and 6, 24 and 8, 25 and 9, 26 and 10, and 28 and 12) and if each of these pairs is merged into a single state (taking the smaller of the state labels as the label for the new state) then the automaton described in Table 3(b) is obtained. Note that, even though all the equivalent pairs of states were merged into single states,

From State	To State	
	−ve	+ve
0	0	1
1	2	3
2	4	5
3	6	7
4	8	9
5	10	11
6	12	13
7	14	15
8	16	17
9	18	19
10	20	21
11	22	23
12	24	25
13	26	27
14	28	29
15	30	⋆
16	0	1
17	2	3
18	4	5
19	6	7
20	8	9
21	10	11
22	12	13
23	14	⋆
24	16	17
25	18	19
26	20	21
27	22	⋆
28	24	25
29	26	⋆
30	28	⋆

(a)

From State	To State	
	−ve	+ve
0	0	1
1	2	3
2	4	5
3	6	7
4	8	9
5	10	11
6	12	13
7	14	15
8	0	1
9	2	3
10	4	5
11	6	23
12	8	9
13	10	27
14	12	29
15	30	⋆
23	14	⋆
27	6	⋆
29	10	⋆
30	12	⋆

(b)

From State	To State	
	−ve	+ve
0	0	1
1	2	3
2	4	5
3	6	7
4	0	1
5	2	11
6	4	13
7	14	15
11	6	23
13	2	27
14	4	29
15	30	⋆
23	14	⋆
27	6	⋆
29	2	⋆
30	4	⋆

(c)

From State	To State	
	−ve	+ve
0	0	1
1	2	3
2	0	5
3	6	7
5	2	11
6	0	13
7	14	15
11	6	23
13	2	27
14	0	29
15	30	⋆
23	14	⋆
27	6	⋆
29	2	⋆
30	0	⋆

(d)

Table 3: State transition tables for 5 out of 6 sliding window detectors

there are still four pairs of equivalent states (namely 8 and 0, 9 and 1, 10 and 2, and 12 and 4) in this new automaton. If the same process is repeated on this new automaton the automaton described in Table 3(c) is obtained and this automaton also contains an equivalent pair (4 and 0) of states. After one more iteration of the process the automaton described in Table 3(d) is obtained and (finally) this is an optimal automaton with no pairs of equivalent states.

If an iterative process based on the above discussion is applied to the set of hexads $\bar{\mathrm{S}}_{M,N}$ produced by Algorithm 3 then the optimal set $\bar{\mathrm{S}}'_{M,N}$ (that which represents an automata for implementing M out of N sliding window detector in the minimal number of states) can be generated. $\bar{\mathrm{S}}'_{M,N}$ can the be substituted for $\bar{\mathrm{S}}_{M,N}$ in Algorithm 2 to provide an optimal algorithm for evaluating M out of N sliding window detectors. The number of hexads in this optimal set is given by

$$\#\bar{\mathrm{S}}'_{M,N} = \begin{cases} \displaystyle\sum_{k=0}^{N-M} \binom{M}{k+1}\binom{N-M}{k} & \text{when } M \leq N \leq 2M-1 \\ \displaystyle\sum_{k=0}^{M-1} \binom{M}{k+1}\binom{N-M}{k} & \text{when } N \geq 2M-1 \end{cases} \tag{39}$$

which in most instances is much smaller than that given by Equation 31 for $\#\bar{\mathrm{S}}_{M,N}$. However, the time taken to generate this optimal set (using this process) will be proportional to $\#\bar{\mathrm{S}}_{M,N}$ rather than $\#\bar{\mathrm{S}}'_{M,N}$ and what is required are versions of Algorithms 3 and 4 that generate the optimal set directly.

Algorithm 5 Create the optimal set of hexads $\bar{\mathrm{S}}'_{M,N}$

$\bar{\mathrm{S}}'_{M,N} = \{(\star, 0, \star, \star, 0, 0)\}$
Add a hexad with $H = 0$ to the optimal set $\bar{\mathrm{S}}'_{M,N}$ – **Algorithm 6**
$\bar{\mathrm{S}}'_{M,N}[0].P \Leftarrow 1$

Algorithms 5 is a modified version of Algorithm 3 which sidesteps the problem by shifting the responsibility to Algorithm 6 which is the modified version of Algorithm 4. The primary modifications that are present in Algorithm 6 are based on the fact that the determination of the values for H^+ and H^- are the key to the problem as they determine which hexads will be created. If an algorithm can be devised that can determine the identifying history H of the hexad that represents (as an equivalent) some other history i then the optimal set can be produced directly.

Algorithm 7 performs this function. It is based on the observation that, when equivalent pairs of hexads (h_1, h_2) occur in $\bar{\mathrm{S}}_{M,N}$ (where the convention of listing pairs in descending order of identifying histories is followed), the member of the pair h_1 with the largest identifying history $h_1.H$ has a component $h_1.H^+$ whose value is not $\star$ and is smaller than the identifying history (i.e. $h_1.H^+ \neq \star$ and $h_1.H^+ < h_1.H$) and that the identifying history $h_2.H$ of the other member of the pair is equal to the integer portion of $h_1.H^+/2$. Therefore, in order to determine the smallest equivalent history to some history i, Algorithm 7 first determines whether i shifts to the accepting state $\star$ after a positive result in

Algorithm 6 Add a hexad with $H = i$ to the optimal set $\bar{\mathrm{S}}'_{M,N}$

create new hexad h
$h.H \Leftarrow i$
$h.P \Leftarrow 0$
if $\#(i) = (M-1)$ **then**
  $h.H^+ \Leftarrow \star$
else
  $n^+ \Leftarrow \left((i \times 2) \bmod 2^{N-1} + 1\right)$
  $h.H^+ \Leftarrow$ (Smallest equivalent history to n^+ in an M of N automaton) – **Algorithm 7**
end if
$n^- \Leftarrow \left((i \times 2) \bmod 2^{N-1}\right)$
$h.H^- \Leftarrow$ (Smallest equivalent history to n^- in an M of N automaton) – **Algorithm 7**
$\bar{\mathrm{S}}'_{M,N} \Leftarrow \bar{\mathrm{S}}'_{M,N} \cup \{h\}$
if a hexad with $H = h.H^+$ does not exist in $\bar{\mathrm{S}}'_{M,N}$ **then**
  Add a hexad with $H = h.H^+$ to the optimal set $\bar{\mathrm{S}}'_{M,N}$ – **Algorithm 6**
end if
if a hexad with $H = h.H^-$ does not exist in $\bar{\mathrm{S}}'_{M,N}$ **then**
  Add a hexad with $H = h.H^-$ to the optimal set $\bar{\mathrm{S}}'_{M,N}$ – **Algorithm 6**
end if

Algorithm 7 Smallest equivalent history to i in an M of N automaton

if $\#(i) = (M-1)$ **then**
  return i
else
  $n \Leftarrow \left((i \times 2) \bmod 2^{N-1} + 1\right)$
  $n' \Leftarrow$ (Smallest equivalent history to n in an M of N automaton) – **Algorithm 7**
  if $n' < i$ **then**
    return the integer portion of $n'/2$
  else
    return i
  end if
end if

which case it is the smallest equivalent state to itself. If this is not the case then the history n of the state to which i would shift on a successful trial in the non-optimal automaton is determined and then a recursive call to Algorithm 7 is made to determine the smallest equivalent history n' to n which is the state which i will shift to in the optimal case. This recursive call provides the equivalent to the exhaustive iterations in the method described above. The value of n' is compared to i and if it is smaller then i will have an equivalent state in the optimal set of states and the history of that equivalent state is returned.

6 Example of Application

Consider the case where a surface ship equipped with an active sonar set (where the sonar operator can be considered a 3 out of 5 sliding window detector) is approaching a submarine and it is required to determine the probability that the ship detects the

submarine before some given range (for example, see ref. 4). In order to solve this problem using the methodology described in this paper the following information is required:

- the probability of detecting the submarine on a single sonar ping as a function of the range between the ship and the submarine,
- the time interval between successive sonar pings, and
- the positions and velocities of the ship and submarine.

For the purposes of this example, the probability of detection on a single ping (as a function of range) will be that illustrated at Figure 3. This is a fictional and stylised version of a situation often encountered. There are two range bands where the probability of detection

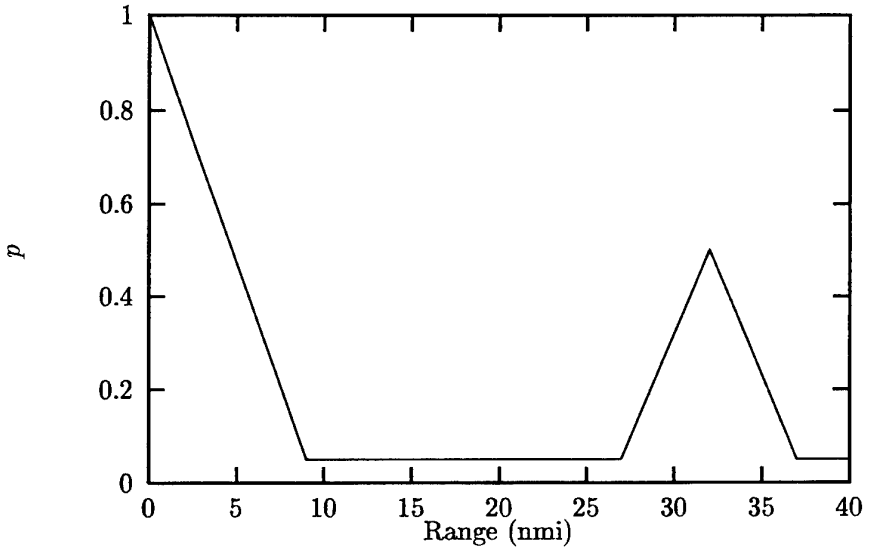

Figure 3: The probability, p, of detection on a single ping as a function of range

is significant:

- a *convergence zone* between (approximately) 27 and 37 nautical miles (nmi), and
- a *direct propagation zone* inside (approximately) 9 nmi.

Outside these two zones the probability of detection is roughly equivalent to the false alarm probability which has been set to (an unrealistically high) 0.05 for the purposes of illustration. In order to maintain a (convenient) linear relationship between time and range it will be further assumed that the ship and submarine are on a collision course with a closing rate of 15 knots. In order to take advantage of the possibility of convergence range detections, the time interval between pings will be set to that which provides an unambiguous maximum range of 40 nmi(approximately 99.4 seconds) with the first ping

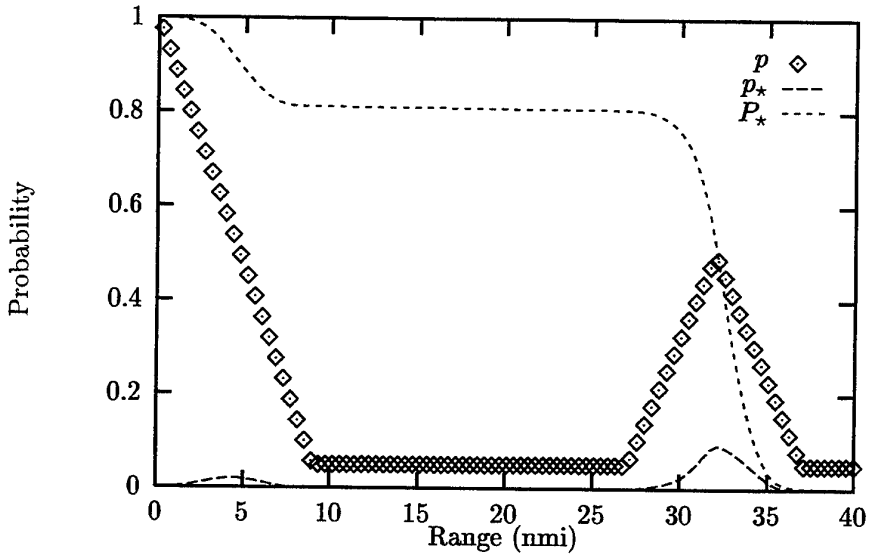

Figure 4: The probabilities, $p_\star$, that satisfaction of the 3 blips in 5 pings criterion is met at a given range and, $P_\star$, that satisfaction of the criterion occurs by a given range

occurring at 40 nmi. The probability ($P_\star$) that the 3 blips out of 5 pings criterion is met *by* a given range (as a function of that range) is shown at Figure 4 along with the probability ($p_\star$) that the satisfaction of the criterion occurs *at* a given range. For the convenience of the reader the samples of the blip/ping ratio used are also shown on the diagram as individual points. It should be noted that $p_\star$ is bimodal.

7 Conclusion

The algorithms described in this paper for evaluating M out of N sliding window detectors are simple and efficient. Their implementation in most computer languages should be simple and straightforward. Applications include modelling of naval surveillance radar and sonar systems.

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19. ABSTRACT

The term *M out of N sliding window detectors* refers to processes that determine whether there have been M successes in a sequence of N binary trials, where the *window* (of N trials under examination) *slides* along a possibly infinite sequence stopping when the criterion (of M successes in N successive trials) is met. They are frequently used to model the operators of naval surveillance systems such as radar and sonar. When an M out of N sliding window detector is examining a sequence of trials, it may be in one of several states. The state of most interest is the *accepting* state into which it enters (and remains) when it encounters a sequence of N trials containing M successes. This paper describes a generalised method for estimating the *probability* that an M out of N sliding window detector is in its accepting state given a sequence of probabilities representing the likelihood of success on each of a sequence of binary trials.